Multipoint scatterers with zero-energy bound states *

P.G. Grinevich † R.G. Novikov‡

Abstract

We study multipoint scatterers with zero-energy bound states in three dimensions. We present examples of such scatterers with multiple zero eigenvalue or with strong multipole localization of zero-energy bound states.

1 Introduction

We consider the model of point scatterers in three dimensions, which goes back to the classical works [4], [6], [9], [3] and presented in detail in the book [1]. For more recent results on such models, see [5], [2], [7] and references therein. More precisely, we consider the stationary Schrödiger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^3, \tag{1}$$

with multipoint potential (scatterer)

$$v(x) = \sum_{j=1}^{n} v_{z_j,\alpha_j}(x), \tag{2}$$

*This work was fulfilled during the visit of the first author to the Centre de Mathématiques Appliquées of École Polytechnique in September-October 2016. The first author was also partially supported by by the program “Fundamental problems of nonlinear dynamics” of the Presidium of RAS.

†L.D. Landau Institute for Theoretical Physics RAS, Chernogolovka, Moscow region, Russia; Moscow State University, Moscow, Russia; Moscow Physical-Technical Institute, Dolgoprudny, Russia; e-mail: pgg@landau.ac.ru

‡CNRS (UMR 7641), Centre de Mathématiques Appliquées, École Polytechnique, Université Paris Saclay, 91128, Palaiseau, France; IEPT RAS, 117997, Moscow, Russia; e-mail: novikov@cmap.polytechnique.fr

consisting of n single-point scatterers $v_{z_j,\alpha_j}(x)$, where each point scatterer $v_{z_j,\alpha_j}(x)$ is described by its position $z_j \in \mathbb{R}^3$ and its internal parameter $\alpha_j \in \mathbb{R}$, where $z_i \neq z_j$ if $i \neq j$.

In the present article we study multipoint scatterers v for which equation (1) admits non-zero solutions $\psi \in L^2(\mathbb{R}^3)$ at energy $E = 0$, or in other words, we study the multipoint scatterers with zero-energy bound states. These studies are motivated, in particular, by studies of low-energy scattering effects in three dimensions. To our knowledge, the question about zero-energy bound states for multipoint scatterers was not considered properly in the literature. Besides, our studies were stimulated by [8], where interesting examples of regular rapidly decaying potentials with well-localized zero energy bound states in two dimensions were constructed using the Moutard transform technique.

Results of the present article include Proposition 1, Theorem 1 and Examples 1 and 2 given below.

2 Solitions of the Schrödinger equation with multipoint potential

We say that ψ satisfies (1) iff

$$-\Delta\psi(x) = E\psi(x) \quad \text{for} \quad x \in \mathbb{R}^3 \backslash \{z_1, z_2, \ldots, z_n\}, \tag{3}$$

and

$$\psi(x) = \frac{\psi_{j,-1}}{|x - z_j|} + \psi_{j,0} + O(|x - z_j|) \quad \text{as} \quad x \to z_j, \quad j = 1, \ldots, n, \tag{4}$$

where

$$\psi_{j,0} = 4\pi\alpha_j\psi_{j,-1}. \tag{5}$$

In this article we use the same normalization of multipoint scatterers as in the book [1], see pages 47, 112.

Proposition 1 *A function $\psi = \psi(x)$ satisfies (3)-(5) if and only if this function admits the following representation:*

$$\psi(x) = \psi_0(x) + \sum_{j=1}^{n} q_j G^+(|x - z_j|, E), \tag{6}$$

where

$$-\Delta\psi_0(x) = E\psi_0(x) \quad \textit{for} \quad x \in \mathbb{R}^3, \tag{7}$$

$$G^+(r,E) = -\frac{e^{i\sqrt{E}r}}{4\pi r}, \quad r > 0, \quad i = \sqrt{-1}, \quad \sqrt{E} \geq 0 \quad \textit{for} \quad E \geq 0, \tag{8}$$

and $\vec{q} = (q_1, \ldots, q_n)^t$ *satisfies the following linear system:*

$$A\vec{q} = \vec{\phi}, \tag{9}$$

where A *is the* $n \times n$ *matrix*

$$A_{j,j'} = \begin{cases} \alpha_j - \frac{i\sqrt{E}}{4\pi} & \textit{for} \quad j' = j \\ G^+(|z_j - z_{j'}|, E) & \textit{for} \quad j' \neq j, \end{cases} \tag{10}$$

and $\vec{\phi} = (\phi_1, \ldots, \phi_n)^t$,

$$\phi_j = -\psi_0(z_j), \quad j = 1, \ldots, n. \tag{11}$$

Proposition 1 is a variation of statements used in the book [1].

3 Zero-energy bound states

Theorem 1 *Equation (1) with multipoint potential* v *of the form (2) admits a non-zero solution* $\psi \in L^2(\mathbb{R}^3)$ *at energy* $E = 0$ *if and only if there exists a non-zero* $\vec{q}$ *such that*

$$A\vec{q} = 0 \quad \textit{for} \quad E = 0, \tag{12}$$

$$\sum_{j=1}^{n} q_j = 0, \tag{13}$$

where A *is defined by (10). In addition, the one-to-one correspondence between such solutions* ψ *and vectors* $\vec{q}$ *is given by:*

$$\psi(x) = -\frac{1}{4\pi}\sum_{j=1}^{n} q_j \frac{1}{|x - z_j|}. \tag{14}$$

Theorem 1 follows from Proposition 1, the property that

$$G^+(|\cdot|, 0) \in L^2_{\mathrm{loc}}(\mathbb{R}^3),$$

the linear independence of $G^+(|\cdot - z_j|, 0)$, $j = 1, \dots, n$, the following asymptotic formula for ψ of the form (14):

$$\psi(x) = -\frac{1}{4\pi|x|}\sum_{j=1}^{n} q_j + O\left(\frac{1}{|x|^2}\right) \quad \text{as} \quad |x| \to +\infty, \tag{15}$$

and the following lemma:

Lemma 1 *Let ψ_0 satisfy (7) for $E = 0$ and:*

$$\psi_0 = \psi_{0,1} + \psi_{0,2}, \quad \psi_{0,1}(x) = o(1) \quad \textit{as} \quad |x| \to \infty, \quad \psi_{0,2} \in L^2(\mathbb{R}^3). \tag{16}$$

Then $\psi_0 \equiv 0$.

Lemma 1 follows from the mean value property over balls for harmonic functions, the Cauchy-Schwarz inequality and Liouville's theorem for harmonic functions.

In the next example we consider a scatterer consisting of four equal single point scatterers located in the vertices of a regular tetrahedron.

Example 1 *Let $n = 4$, $z_j \in \mathbb{R}^3$, $|z_j - z_{j'}| = s > 0$ for all $j \neq j'$, $1 \leq j, j' \leq 4$, $\alpha_j = \alpha = -(4\pi s)^{-1}$, and v be given by (2). Then $E = 0$ is a triple eigenvalue for equation (1).*

This statement follows directly from Theorem 1.

In the next example we consider a scatterer consisting of $2m$ equal single point scatterers located in the vertices of a regular planar $2m$-gon.

Example 2 *Let $n = 2m$, $z_1, \dots, z_{2m} \in \mathbb{R}^3$ be sequentially enumerated vertices of a convex regular planar (belonging to a fixed plane) polygon with $2m$ vertices,*

$$\alpha_j = \alpha = -\sum_{k=2}^{2m} \frac{(-1)^k}{4\pi|z_k - z_1|}, \tag{17}$$

and v be given by (2). Then:

$$\alpha \neq 0, \tag{18}$$

$$\psi(x) = -\frac{1}{4\pi}\sum_{j=1}^{2m} \frac{(-1)^{j+1}}{|x - z_j|} \tag{19}$$

is a zero-energy bound state for equation (1);

$$\psi(x) = O\left(\frac{1}{|x|^{m+1}}\right) \quad as \quad |x| \to +\infty. \tag{20}$$

The point is that in this example the zero-energy bound state ψ is strongly localized for large m.

In addition, we have the conjecture that $E = 0$ is a simple eigenvalue in this example; it was checked numerically up to $m = 48$ using Theorem 1.

The property (18) follows from the formulas:

$$\alpha = -\sum_{k=2}^{m+1}(-1)^k u_k, \quad u_k = \begin{cases} \frac{1}{2\pi|z_k - z_1|}, & k = 2, \ldots, m, \\ \frac{1}{4\pi|z_{m+1} - z_1|}, & k = m+1, \end{cases} \tag{21}$$

$$u_2 > u_3 > \ldots > u_{m+1} > 0. \tag{22}$$

Formulas (17), (19) were obtained using (12) with $q_j = (-1)^{j+1}$, $j = 1, \ldots, n$, and finding α such that (12) holds for $\alpha_j = \alpha$, $j = 1, \ldots, n$.

To prove the localization property (20) we choose orthogonal coordinates such that

$$z_j = r_0\,\omega_j, \quad r_0 > 0, \quad \omega_j = \left(\cos\left(\frac{\pi(j-1)}{m}\right), \sin\left(\frac{\pi(j-1)}{m}\right), 0\right), \qquad j = 1, \ldots, 2m. \tag{23}$$

We have

$$\frac{1}{|x - z_j|} = \frac{1}{(R^2 + r_0^2)^{1/2}} \sum_{l=0}^{+\infty} b_l \left(\frac{2r_0 R}{R^2 + r_0^2}\right)^l (\nu\omega_j)^l, \quad R \to \infty, \tag{24}$$

where

$$R = |x|, \quad \nu = x/|x|, \quad \nu = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$

θ, ϕ are the polar and azimuthal angles of ν, respectively, b_l are the expansion coefficients:

$$(1 - t)^{-1/2} = \sum_{l=0}^{+\infty} b_l t^l, \quad |t| < 1. \tag{25}$$

Thus,

$$\psi(x) = \frac{1}{4\pi}\frac{1}{(R^2+r_0^2)^{1/2}}\sum_{l=0}^{+\infty} b_l\left(\frac{2r_0R}{R^2+r_0^2}\right)^l \left[\sum_{j=1}^{2m}(-1)^j(\nu\omega_j)^l\right], \quad R\to\infty, \tag{26}$$

$$\nu\omega_j = \sin\theta\cos\left(\phi - \frac{\pi(j-1)}{m}\right).$$

The localization (20) follows from the property:

$$\mathcal{C}_l := \sum_{j=1}^{2m}(-1)^j(\nu\omega_j)^l = 0 \quad \text{for} \quad 0\le l\le m-1. \tag{27}$$

In turn, identity (27) follows from the formulas:

$$\mathcal{C}_l = \mathcal{C}_l(\theta,\phi) = (\sin\theta)^l \sum_{k=-l}^{l} c_{lk}e^{ik\phi} \quad \text{for some} \quad c_{lk} \quad \text{depending on} \quad m; \tag{28}$$

$$\mathcal{C}_l(\theta,\phi+\pi/m) = -\mathcal{C}_l(\theta,\phi). \tag{29}$$

This completes the proof of Example 2.

References

[1] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, H. Holden, Solvable models in quantum mechanics, Texts and Monographs in Physics. Springer-Verlag, New York, 1988; doi:10.1007/978-3-642-88201-2.

[2] N.P. Badalyan, V.A. Burov, S.A.Morozov, O.D. Rumyantseva, Scattering by acoustic boundary scattering with small wave sizes and their reconstruction, *Acoustical Physics* **55:1** (2009), 1–7; doi:10.1134/S1063771009010011.

[3] F.A. Berezin and L.D. Faddeev, Remark on Schrödinger equation with singular potential, *Soviet Mathematics* **2** (1961), 372–375.

[4] H. Bethe and R. Peierls, Quantum Theory of the Diplon, *Proc. R. Soc. Lond. A*, **148** (1935), 146–156; doi:10.1098/rspa.1935.0010.

[5] G. Dell'Antonio, R. Figari, A. Teta, A brief review on point interactions, *Inverse problems and imaging. Lecture Notes in Math.*, **1943**, 171–189 Springer, Berlin, 2008; doi:10.1007/978-3-540-78547-7

[6] E. Fermi, Sul moto dei neutroni nelle sostanze idrogenate, *La Ricerca Scientifica*, **7**(2) (1936), 13–52 (F. N. M. vol. I, 943–979).

[7] P.G. Grinevich, R.G. Novikov, Faddeev eigenfunctions for multipoint potentials, *Eurasian Journal of Mathematical and Computer Applications*, **1**:2 (2013), 76–91.

[8] I.A. Taimanov, S.P. Tsarev, Two-dimensional Schrödinger operators with fast decaying potential and multidimensional L_2-kernel, *Russian Mathematical Surveys*, **62**:3 (2007), 631–633; doi:10.1070/RM2007v062n03ABEH004423.

[9] Ya.B. Zel'dovich, Scattering by a singular potential in perturbation theory and in the momentum representation, *Soviet Physics. JETP*, **11**:3 (1960), 594–597.